J. Fluid Mech. (1973), *vol.* 58, *part* 1, *pp.* 65–80
Printed in Great Britain

Radiation from line vortex filaments exhausting from a two-dimensional semi-infinite duct

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(Received 4 December 1972)

The low Mach number sound field induced by the motion of line vortex filaments coupled to a two-dimensional semi-infinite duct is determined by means of a singular perturbation technique. Using the method of matched asymptotic expansions, solutions for the sound field are obtained by matching with an 'inner' region of incompressible flow. The radiation field induced by the emergence of a single vortex from the channel exhibits the edge scattering effects typical of half-plane problems. The sound field intensity is found to have angular dependence on $\sin^2 \frac{1}{2}\theta$, where $\theta = 0$ defines the exterior axis of symmetry. When a vortex pair propagates out of the duct however, the special symmetry of the fluid motion causes cancellation of the scattered field from the duct edges. In that case the sound field is driven from sources located at the duct exit. We show that this result is consistent with the general theories of both Curle and Powell. The sound field is essentially induced by a dipole at the exit plane of the duct, part of which drives a coupled weak monopole, while the remainder corresponds to an axial 'edge force' originating in the $r^{-\frac{1}{2}}$ velocity singularities at the duct edges.

1. Introduction

The work which follows was motivated by an interest in sound radiation in the presence of solid surfaces, from fluid flows containing concentrated vorticity. In general, two-dimensional, unsteady, compressible, rotational flows are not amenable to analysis, and few exact solutions have been found. Stüber (1970) and Rahman (1971) have considered the sound field generated by a vortex pair near an infinite rigid plane. The surface effect in this case is essentially trivial, resulting in a reflexion of the sound field by the plane (Powell 1960). The general scattering theories developed by Ffowcs Williams & Hall (1970), Crighton & Leppington (1970, 1971) and Jones (1972) predict powerful and interesting effects when flows are coupled to large inhomogeneous solid surfaces. Recent attention has been concentrated on geometries involving edge scattering, and Crighton (1972) has solved the problem of sound generation by a line vortex interacting with a semi-infinite rigid plane. His results confirm the prediction of general scattering theory that sharp edges support powerful sources of sound and he determines the source strengths. This paper adopts Crighton's analytic scheme to study the sound generated by vortices exhausting from a two-dimensional semi-infinite duct. The vortex axes are parallel to the duct edges at all times, and the problem of determining the acoustic field is posed in a particularly simple way. Providing that the vortices are sufficiently weak, and the circulations of the correct sense, the vortex system will propagate out of the channel with a speed which is always small in comparison with c, the speed of sound. The path followed by each vortex is virtually straight, except near the mouth of the channel, and because of this the effective 'source' region is confined to the compact vicinity of the channel opening. Radiation occurs only while the vortices travel on curved paths near the mouth of the duct. Thus it is appropriate to consider the problem as one of singular perturbation type, the small parameter M being the ratio of L, the length scale of the unsteady vortex motion, to a characteristic acoustic wavelength λ, set by the vortex propagation velocity. The acoustic field is determined by matching an 'inner' incompressible field scaling on L to a wave field scaled on the acoustic wavelength. The 'inner' field is determined by conformal transformation.

In §§ 2 and 3 we consider the motion of a vortex pair coupled to a semi-infinite duct. The unsteady flow field is symmetric with respect to the duct axis, and as a result edge scattering effects cancel. We find a solution for the sound field that is acoustically equivalent to compact dipole and weak monopole sources acting at the duct exit, the dipole term constituting a force acting in the axial direction $\theta = 0$. Ffowcs Williams & Gordon (1964) pointed out the likelihood of there being both monopole and dipole exit plane sources, and recent work (Ffowcs Williams *et al.* 1972) has drawn attention to the interdependence of these sources.

Since edge scattering effects are weak we are able to apply Curle's (1955) theory, in which he extended the results of Lighthill's original papers on aerodynamic sound (1952, 1954) to include the effect of solid surfaces. Using the symmetry properties of the flow, we find that Curle's results predict an acoustic field generated by a distribution of sources on the exit plane, which is equivalent to a compact dipole source with directionality $\cos\theta$. Such a force would drive an induced mass flow from the duct interior, and we would expect the dipole and monopole strengths to be coupled. The coupling, however, is not straightforward since there exists on the sharp edges of the duct a force producing singularity in the pressure field. We find that this force generates an axial dipole which is additional to that supported by the unsteady thrust at the exit plane. The thrust-generated dipole is simply connected to a monopole representing the volume response of the interior channel flow. These coupled sources generate a field that vanishes at $\theta = 0$. The behaviour is consistent with the view that the solid duct surfaces support dipoles with axes normal to the duct axis. Only an axial force, supported by an edge singularity, can generate finite dipole sound on the duct axis. We find the strength of this edge force to be equal to the vortex source strength given by Powell (1964). Powell's source, $\rho_0 \boldsymbol{\omega} \times \mathbf{u}$ per unit volume, is therefore only a part (but a recognizable part) of the full source field. The sound field can therefore be regarded as driven (i) by pressure forces acting across a free surface within the duct which are coupled to an induced mass flux, and (ii) by a volume distribution of dipole sources of strength $\rho_0 \boldsymbol{\omega} \times \mathbf{u}$. Since $\boldsymbol{\omega}$ may be represented by delta-function singularities at the vortex centres the vector $\boldsymbol{\omega} \times \mathbf{u}$ takes a particularly simple form in our problem. The essential results of §§ 2–4 are: first, the determination of the dipole and monopole source strengths in terms of the vortex parameters, and second, recognition of precisely how the various source elements are inter-related.

In § 5 we determine the motion of a single vortex exhausting from a duct, and in this case, since the unsteady motion is no longer symmetric about the duct axis, the normal efficient edge scattering is recovered. The sound field is qualitatively equivalent to that obtained from edge scattering by a half-plane in two dimensions, and has an angular dependence of $\sin^2 \frac{1}{2}\theta$ on intensity. We show how the source strengths can be calculated, and compare the efficiency of the duct as a radiator of energy into the sound field with the efficiency of a half-plane, using the results of Crighton's (1972) paper.

2. Incompressible flow

To gain insight into the length scales involved in the motion of a vortex pair coupled to a semi-infinite duct we suppose that the fluid is incompressible.

Consider two-dimensional motion in a complex-ζ plane with co-ordinates defined by $\zeta = \xi + i\eta$. Let $\xi + i\pi$ and $\xi - i\pi$, where $-\infty < \xi < +\infty$, define the walls of an infinite channel in this plane. We determine the instantaneous complex potential due to a negative line vortex at

$$\zeta_0 = \xi_0 + i(\pi - \delta), \quad \delta < \pi, \tag{2.1}$$

and a positive line vortex of equal strength K at the complex conjugate point $\bar{\zeta}_0$. The action of the unbounded channel walls can be represented by an infinite image system, and by summing the velocity induced at a point ζ by all the vortices, we obtain an equation for the complex velocity from which the complex potential $\Omega(\zeta)$ can be deduced:

$$\frac{d\Omega}{d\zeta} = \frac{-K}{4\pi}\{\cot \tfrac{1}{2}i(\zeta - \zeta_0) - \cot \tfrac{1}{2}i(\zeta - \bar{\zeta}_0)\}. \tag{2.2}$$

Integrating (2.2) with respect to ζ we find that the complex potential can be written as

$$\Omega(\zeta) = \phi + i\psi = \frac{iK}{2\pi}\log_e\left\{\frac{\sin \frac{1}{2}i(\zeta - \zeta_0)}{\sin \frac{1}{2}i(\zeta - \bar{\zeta}_0)}\right\} + \gamma(t), \tag{2.3}$$

where $\gamma(t)$ is an arbitrary function of time.

The vortex pair maintains its symmetry with respect to the ξ axis, and the image system as a whole remains on a line parallel to $O\eta$ which propagates with constant velocity in the ξ direction. When $\delta = \frac{1}{2}\pi$, the total velocity induced by all the members of the image system at any individual vortex is zero, and according as $\delta \gtrless \frac{1}{2}\pi$ the propagation velocity is greater or less than zero.

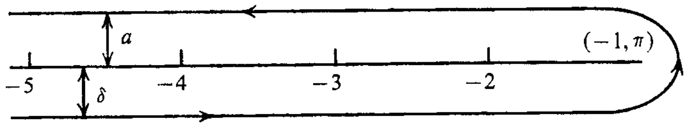

FIGURE 1. Path of vortex near upper plate: $\delta = 0{\cdot}34$.

Let co-ordinates in the complex-Z plane be represented by $Z = x+iy$. The conformal mapping

$$Z = e^{\zeta}+\zeta \tag{2.4}$$

maps the infinite channel in the ζ plane to a semi-infinite channel in the Z plane with walls defined by

$$Z = x \pm i\pi, \quad -\infty < x \leqslant -1. \tag{2.5}$$

Let $f(Z)$ be an implicit function such that

$$\zeta = f(Z), \tag{2.6}$$

then the complex velocity in the Z plane is

$$\frac{d\Omega}{dZ} = u-iv = \frac{K}{4\pi}\left(\frac{-1}{1+e^{f(Z)}}\right)\{\cot \tfrac{1}{2}i(f(Z)-f(Z_0)) - \cot \tfrac{1}{2}i(f(Z)-\overline{f(Z_0)})\}, \tag{2.7}$$

and the complex potential is

$$\Omega(Z) = \phi + i\psi = \frac{iK}{2\pi}\log_e\left\{\frac{\sin \frac{1}{2}i(f(Z)-f(Z_0))}{\sin \frac{1}{2}i(f(Z)-\overline{f(Z_0)})}\right\}+\gamma(t). \tag{2.8}$$

To determine the paths followed by the vortices, we evaluate the complex velocity at $Z_0+\epsilon$ (or $\bar{Z}_0+\epsilon$), where $|\epsilon| \leqslant 1$. The $O(1)$ term in the expansion of $d\Omega/dZ$ in increasing powers of ϵ gives the instantaneous velocity of the vortex at Z_0 (or $\bar{Z}_0$). We obtain

$$u-iv|_{Z_0} = \frac{dx_0}{dt} - i\frac{dy_0}{dt} = -\frac{K}{4\pi}\left\{\frac{1}{1+e^{f(Z_0)}}\right\}\left\{\cot \operatorname{Im} f(Z_0) + \frac{i\, e^{f(Z_0)}}{1+e^{f(Z_0)}}\right\}. \tag{2.9}$$

Since $f(Z_0)$ is only known in implicit form we have to determine the vortex paths by successive numerical integrations, at each step solving for $f(Z_0)$ given the value of Z_0. We have computed vortex paths for various values of δ, and figure 1 illustrates a typical result when $\delta < \frac{1}{2}\pi$. In general the paths are asymmetric with respect to the nearest duct wall, although as $\delta \to 0$, $a \to \delta$. The maximum speed is achieved near $t = 0$ and is approximately twice $(K/4\pi)\cot\delta$, the asymptotic value of the propagation velocity deep inside the duct. Outside the duct as $x \to -\infty$, the vortex speed tends to

$$\frac{K}{4\pi}\left(\frac{1}{\operatorname{Im} Z_0 - \operatorname{Im} f(Z_0)}\right) \sim \frac{K}{4\pi a}.$$

When $\delta > \frac{1}{2}\pi$ the vortices still travel on curved paths near the duct exit, but in this case they deviate towards the duct axis and then propagate on in a straight line to $x = +\infty$. As $\delta \to \pi$, the limit of the vortex paths is the straight line $y = 0$.

We conclude that the vortex motion has a length scale, L say, which is of the same order of magnitude as the duct width and a velocity scale U which depends on both L and the vortex strength K. In the next section we suppose that the fluid is slightly compressible and use these results to formulate a singular perturbation problem for the sound field.

3. Sound field induced by a vortex pair

Suppose that U and L denote the velocity and length scales of the vortex motion. We assume now that the fluid is slightly compressible in the sense that the Mach number $M = U/c \ll 1$ (c is the speed of sound). Appreciable sound radiation only occurs while the vortex pair is near the mouth of the duct, so we can characterize the period during which radiation is significant by $T = L/U$. During time T, sound travels a distance $cT = LM^{-1}$, and we assume that this distance is much greater than the width of the channel, as indeed it must be for sufficiently weak vortices. We introduce 'outer' co-ordinates (x', y') scaled on the acoustic wavelength LM^{-1}, and 'inner' co-ordinates (X, Y) scaled on L; the systems are related by

$$(x', y') = (MX, MY). \tag{3.1}$$

In addition, we introduce non-dimensional time and frequency variables τ and ϖ, such that

$$\tau = T^{-1}t, \quad \varpi = T\omega, \tag{3.2}$$

and employ the following notation:

$$\left.\begin{aligned} x' + iy' &= r\,e^{i\theta}, \\ Z = X + iY &= R\,e^{i\theta} \end{aligned}\right\} \tag{3.3}$$

((r, θ) and (R, θ) are polar co-ordinates).

The singular perturbation problem can now be posed by looking for the appropriate form of the equations of motion in two distinct limits of small Mach number. The implications of the two sets of dimensionless co-ordinates defined above have been examined by Obermeier (1967) and Rahman (1971), and here we simply state the results. In the limit as $M \to 0$ for fixed (X, Y), the velocity potential ϕ satisfies Laplace's equation

$$\partial^2\phi/\partial X^2 + \partial^2\phi/\partial Y^2 = 0 \tag{3.4}$$

everywhere exterior to the vortex centres. On the other hand, for fixed (x', y') and $M \to 0$, the Navier–Stokes equations reduce to a homogeneous wave equation for the 'outer' velocity potential ϕ_0:

$$\frac{\partial^2\phi_0}{\partial\tau^2} - \frac{\partial^2\phi_0}{\partial x'^2} - \frac{\partial^2\phi_0}{\partial y'^2} = 0. \tag{3.5}$$

If the fluid were truly incompressible there would be no induced flow from the duct. However, since we are considering slightly compressible flow we cannot exclude the possibility that the sound field interacts with the channel to produce an order M mass flux. We represent this unknown flow by the complex potential

$MV\zeta$ in the complex-ζ plane. Including this extra term, we obtain as solution to the 'inner' problem a modified form of (2.8):

$$\phi = \operatorname{Re}\Omega(Z) = \frac{iK}{2\pi}\log_e\left\{\frac{\sin\frac{1}{2}i(f(Z)-f(Z_0))}{\sin\frac{1}{2}i(f(Z)-\overline{f(Z_0)})}\right\} + LV(\tau)f(Z)+\gamma(\tau). \tag{3.6}$$

To obtain solutions for the 'outer' potential valid inside and outside the duct we must match the 'outer' asymptotics of the incompressible field to a one-dimensional wave in the interior of the duct and a two-dimensional wave field exterior to the duct.

Performing the interior matching first, we write the 'inner' potential ϕ in terms of $X = x'/M$, and determine the limit as $M \to 0$ in such a way that $X \to -\infty$ with Y fixed inside the duct. For large negative X we can write

$$f(Z) = Z. \tag{3.7}$$

The 'outer' asymptotics of the 'inner' field are

$$\phi(x'/M, \tau) = \gamma(\tau) - (K/2\pi)\,\eta_0(\tau) + LV(\tau)\,x', \tag{3.8}$$

the remaining terms being exponentially small. ($Z_0 = X_0 + iY_0$ and $f(Z_0) = \xi_0 + i\eta_0$ are used to represent the vortex position and its image in the ζ plane in non-dimensional form.) The 'outer' potential ϕ_0 must match the 'inner' solution as $x' \to 0$ and satisfy a radiation condition as $x' \to -\infty$. Applying the former condition, ϕ in (3.8) must satisfy

$$\partial\phi/\partial\tau = \partial\phi/\partial x' \tag{3.9}$$

and we obtain a relation between γ, η_0 and V such that

$$V(\tau) = \frac{1}{L}\frac{\partial}{\partial\tau}\left(\frac{K}{2\pi}\eta_0(\tau) - \gamma(\tau)\right). \tag{3.10}$$

Outside the duct, as $|Z| \to \infty$, we can write

$$f(Z) = \log_e Z. \tag{3.11}$$

Writing $\Omega(Z)$ in 'outer' variables and letting $M \to 0$ we obtain on taking the real part of Ω

$$\phi = \gamma(\tau) + \frac{K}{2\pi}\eta_0(\tau) + \frac{MK}{2\pi}\frac{g(\tau)\cos\theta}{r} + MLV(\tau)\log r + O(M^2), \tag{3.12}$$

where
$$g(\tau) = 2(Y_0(\tau) - \eta_0(\tau)). \tag{3.13}$$

The 'outer' potential ϕ_0 is a solution of the two-dimensional wave equation (3.5), and as $r \to 0$, ϕ_0 must match the 'outer' asymptotics of ϕ given by (3.12). In addition ϕ_0 must satisfy a hard-wall boundary condition

$$\partial\phi_0/\partial y' = 0 \quad \text{on} \quad \theta = \pm\pi \tag{3.14}$$

and an outgoing wave radiation condition as $r \to \infty$. An immediate consequence of the matching and radiation conditions is that

$$\gamma(\tau) + (K/2\pi)\,\eta_0(\tau) = 0. \tag{3.15}$$

On substituting for γ in (3.10) we obtain an expression for the induced flow in terms of the parameters of the vortex motion;

$$V = -\frac{K}{\pi L}\frac{\partial \eta_0}{\partial \tau}. \tag{3.16}$$

To solve for ϕ_0 in (3.5), we take Fourier transforms in time, denoting transformed quantities with an asterisk, viz:

$$\phi_0(\mathbf{x}',\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_0^*(\mathbf{x}',\varpi)\, e^{-i\varpi\tau}\, d\varpi. \tag{3.17}$$

Multiplying each term of (3.5) by $e^{i\varpi\tau}$ and integrating over all values of τ, we find that ϕ_0^* is a solution of the homogeneous Helmholtz equation

$$\frac{\partial^2\phi_0^*}{\partial x'^2} + \frac{\partial^2\phi_0^*}{\partial y'^2} + \varpi^2\phi_0^* = 0. \tag{3.18}$$

We seek solutions of the form

$$\phi_0^* = \alpha^*(\mathbf{x}',\varpi) + \beta^*(\mathbf{x}',\varpi)\cos\theta \tag{3.19}$$

subject to the appropriate transformed boundary conditions. The matching condition for ϕ_0^* becomes

$$\left.\begin{aligned} \lim_{M\to 0} \alpha^*(MX, MY, \varpi) &= M\int_{-\infty}^{\infty} V(\tau)\, e^{i\varpi\tau}\, d\tau \log R, \\ \lim_{M\to 0} \beta^*(MX, MY, \varpi) &= \frac{1}{R}\int_{-\infty}^{\infty} g(\tau)\, e^{i\varpi\tau}\, d\tau. \end{aligned}\right\} \tag{3.20}$$

For positive frequencies the only solutions of (3.18) which can satisfy all these conditions are

$$\left.\begin{aligned} \alpha^* &= A(\varpi)\, H_0^{(1)}(\varpi r), \\ \beta^* &= B(\varpi)\, H_1^{(1)}(\varpi r). \end{aligned}\right\} \tag{3.21}$$

$H_0^{(1)}$ and $H_1^{(1)}$ denote Hankel functions of the first kind and of the zeroth and first order respectively.

For $\varpi < 0$, we use the fact that

$$\phi_0^*(\mathbf{x}', -\varpi) = \overline{\phi_0^*(\mathbf{x}',\omega)} \quad (\omega > 0), \tag{3.22}$$

the overbar denoting complex conjugation. On applying the matching condition to α^* and β^*, we find that

$$A(\varpi) = \frac{-i\pi}{2} LM \int_{-\infty}^{\infty} V(\tau)\, e^{i\varpi\tau}\, d\tau \tag{3.23}$$

and

$$B(\varpi) = \frac{i\pi}{2} M\varpi \int_{-\infty}^{\infty} g(\tau)\, e^{i\varpi\tau}\, d\tau. \tag{3.24}$$

Thus we have a solution to the 'outer' problem

$$\phi_0^*(\mathbf{x}',\varpi) = A(\varpi)\, H_0^{(1)}(\varpi r) + B(\varpi)\, H_1^{(1)}(\varpi r)\cos\theta \tag{3.25}$$

in terms of known functions η_0 and g which are related to the unsteady vortex motion. Functions of position and time can in principle be obtained from (3.25) by performing the inverse Fourier transform, although to attempt this numerically would involve numerical computations of excessive length.

As an alternative to working in frequency space, we can recognize directly that (3.12) may be identified with the 'inner' asymptotics of the sound field generated by compact dipole and weak monopole sources acting at the origin. The monopole strength per unit volume is

$$Q(\tau) = \frac{-2MK\rho_0}{L^2}\frac{\partial \eta_0}{\partial \tau}, \tag{3.26}$$

while the dipole strength per unit volume is

$$F(\tau) = \frac{2\rho_0 KU}{L^2}\left(\frac{\partial Y_0}{\partial \tau} - \frac{\partial \eta_0}{\partial \tau}\right), \tag{3.27}$$

acting in the positive x direction. The wave field density ρ satisfies

$$\frac{\partial^2 \rho}{\partial t^2} - c^2\frac{\partial^2 \rho}{\partial x^2} - c^2\frac{\partial^2 \rho}{\partial y^2} = \frac{\partial Q(t)}{\partial t}\delta(x_i) - \frac{\partial}{\partial x}\{F(t)\,(x_i)\}. \tag{3.28}$$

By writing (3.28) in non-dimensional space–time variables and solving, we obtain

$$\rho(\mathrm{x}', \tau) = \frac{M^3K\rho_0}{2\pi LU}\int_{-\infty}^{\tau - r}\frac{\partial^2\eta_0(\tau')}{\partial \tau'^2}\frac{d\tau'}{\{(\tau-\tau')^2 - r^2\}^{\frac{1}{2}}} + \frac{M^3K\rho_0}{2\pi LU}\frac{\partial}{\partial x'}\int_{-\infty}^{\tau - r}\frac{\partial g(\tau')}{\partial \tau'}\frac{d\tau'}{\{(\tau-\tau')^2 - r^2\}^{\frac{1}{2}}}. \tag{3.29}$$

Figure 2 indicates how the functions $\partial g/\partial \tau$ and $\partial \eta_0/\partial \tau$ vary with time for a typical vortex configuration and shows clearly how the unsteady variation of the 'outer' asymptotics of the incompressible field is concentrated into the period while the vortex is near the duct exit.

With the help of some algebraic manipulation, and the application of results from the theory of generalized functions (Lighthill 1958) we can derive a formula for E, the total energy per unit length radiated to infinity in the sound field throughout all time:

$$E = -\frac{\rho_0 M^2K^2}{\pi}\int_{-\infty}^{\infty}\tau\log|\tau|\,\frac{d^3F}{d\tau^3}\,d\tau = \frac{\mu\rho_0 M^2K^2}{\pi}, \tag{3.30}$$

where

$$F(\tau) = \int_{-\infty}^{\infty}\left\{\frac{\partial \eta_0}{\partial S}\frac{\partial \eta_0(S+\tau)}{\partial S} + \frac{1}{2}\frac{\partial g}{\partial S}\frac{\partial g(S+\tau)}{\partial S}\right\}dS. \tag{3.31}$$

Since $M = U/c$ and the velocity scale U is proportional to the vortex strength K, E varies as the fourth power of K (or the velocity scale U). This is characteristic of a two-dimensional dipole (Ffowcs Williams 1969). Numerical calculations were carried out to determine the order of magnitude of the non-dimensional coefficient μ, for various values of δ. The results show that μ is 1500 when $\delta = 0{\cdot}1$ and decreases monotonically to approximately 30 when $\delta = 1{\cdot}45$. Over the range $0 < \delta < \frac{1}{2}\pi$, the mean value of μ is approximately 270. For $\frac{1}{2}\pi < \delta < \pi$, changes in the unsteady hydrodynamic field are smaller, and as a result the duct radiates energy into the sound field less efficiently; the mean value of μ in this case is

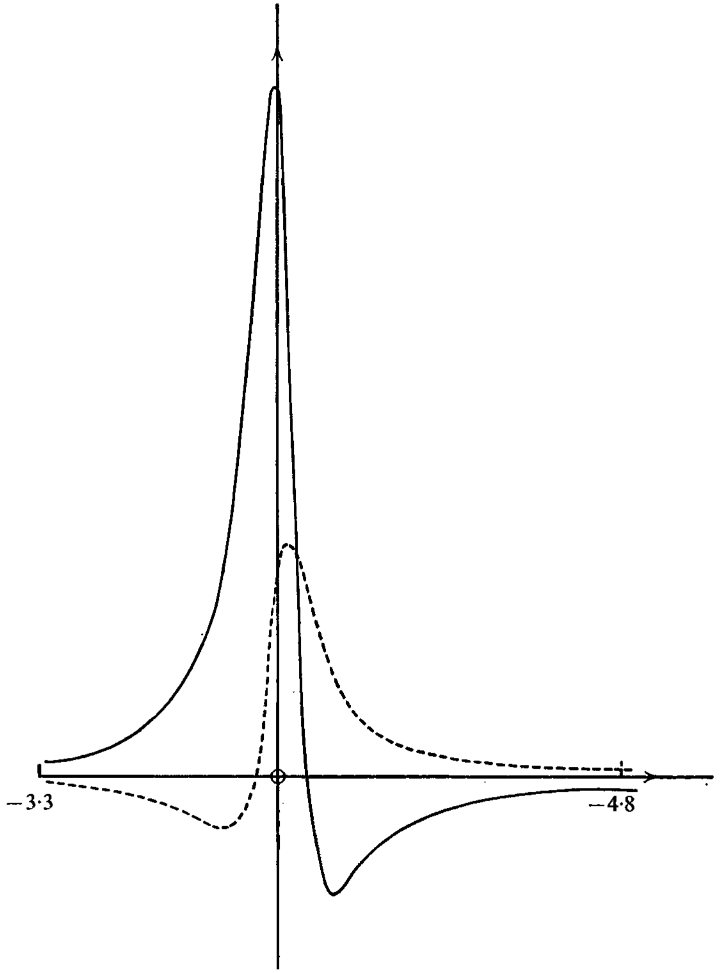

FIGURE 2. Variation of $\partial g/\partial \tau$ (solid line) and $\partial \eta_0/\partial \tau$ (broken line) with time: $\delta = 0{\cdot}65$. The figures $-3{\cdot}3$ and $-4{\cdot}8$ refer to the value of x inside and outside the duct respectively.

about 1·0. In § 4 we check these results obtained by matching with the theoretical predictions of Curle (1955) and Powell (1964), and deduce why the monopole and dipole sources are coupled in this particular way.

4. Solutions to Lighthill's equation

The work which follows does not strictly speaking constitute an independent check on the solution of § 3 since we use the value obtained in that section for the induced velocity V; nevertheless, it gives a useful insight into the origin of the sources which generate the sound field. We use the familiar idea in aerodynamic sound problems of separating the flow field into distinct generation and propagation regions. The effective compact source region is assumed to be bounded near the origin by the surface S_0+S_1 of figure 3, on and near which the various flow parameters can be determined from the 'incompressible' analysis of §§ 2 and 3.

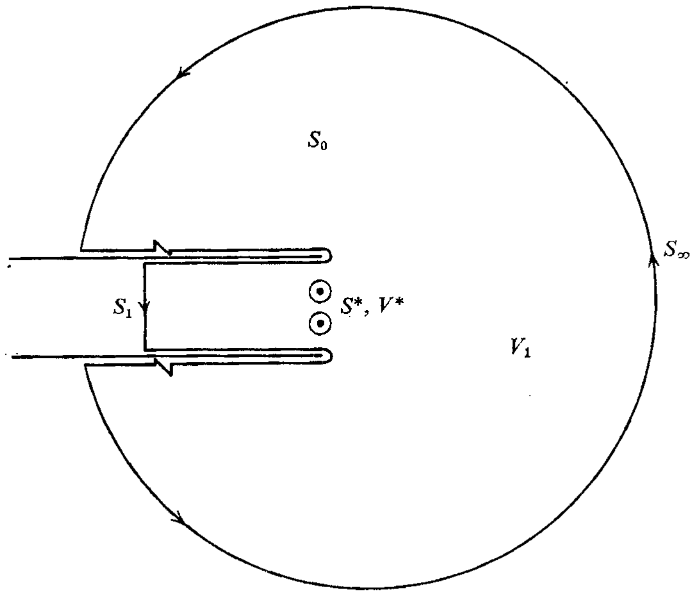

FIGURE 3. $S = S_0 + S_1 + S_\infty$.

The exact equations of motion for an inviscid fluid can be written as

$$\partial\rho/\partial t + \partial(\rho u_i)/\partial x_i = 0, \tag{4.1}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial x_i} = 0 \tag{4.2}$$

(p is the pressure, ρ the density, $\mathbf{u} = (u_1, u_2, u_3)$ the velocity and $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$ is the vorticity vector).

Lighthill (1952) showed how the two equations can be combined to form an inhomogeneous wave equation for ρ:

$$\frac{\partial^2\rho}{\partial t^2} - c^2\frac{\partial^2\rho}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i\,\partial x_j}, \tag{4.3}$$

where
$$T_{ij} = \rho u_i u_j + (p - c^2\rho)\,\delta_{ij}. \tag{4.4}$$

If V' is the volume of fluid inside S, then Curle (1955) obtained a general solution to (4.3) in the presence of S which can be written as

$$\rho(\mathbf{x},t) - \rho_0 = \frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i\partial x_j}\int_{V'}\frac{[T_{ij}]}{r}\,d\mathbf{y} + \frac{1}{4\pi c^2}\frac{\partial}{\partial x_i}\int_S l_j\frac{1}{r}[p\delta_{ij} + \rho u_i u_j]\,dS\,(\mathbf{y})$$
$$- \frac{1}{4\pi c^2}\int_S l_i\left[\frac{1}{r}\frac{\partial}{\partial t}(\rho u_i)\right]dS(\mathbf{y}) \tag{4.5}$$

($r = |\mathbf{x} - \mathbf{y}|$, $[f] \equiv f(\mathbf{y}, t - r/c)$ and $\mathbf{n} = (l_1, l_2, l_3)$ is the unit outward normal from the fluid). For convenience we present the solution in dimensional variables and as integrals over three space co-ordinates (y_1, y_2, y_3). Since the only dependence on y_3 arises from the retarded times the y_3 integration is effectively over time. By a suitable co-ordinate transformation (Ffowcs Williams & Hawkins 1968) we can recover the usual form of solutions to the two-dimensional wave equation (cf. equation (3.29)).

The first term of (4.5) represents a volume distribution of quadrupole sources and can be neglected to $O(M^2)$ compared with the surface integrals. We can also neglect density variations in the compact sound-producing region. The surface integrals vanish identically on the distant surface S_∞. In addition, if the surface S_1 is positioned far away from the source region, but close to the exit on a wavelength scale, the conditions there can be represented by their asymptotic values in the hydrodynamic far field. Equation (4.5) can then be simplified to give

$$\rho-\rho_0 = \frac{1}{4\pi c^2}\frac{x_i}{|\mathbf{x}|^2}\frac{\partial}{\partial t}\int_{S_1} l_i p\,dS + \frac{1}{4\pi c^2}\frac{x_i}{|\mathbf{x}|^2}\frac{\partial}{\partial t}\int_{S_0} l_j(p\delta_{ij}+\rho_0 u_i u_j)\,dS - \frac{1}{4\pi c^2}\frac{1}{|\mathbf{x}|}\frac{\partial}{\partial t}\int_{S_1} l_i\rho_0 u_i\,dS. \quad (4.6)$$

The boundary S_0 consists of streamlines around each wall of the duct, so that $l_j u_j = 0$ and

$$\rho-\rho_0 = \frac{1}{4\pi c^2}\frac{x_i}{|\mathbf{x}|^2}\frac{\partial}{\partial t}\int_{S_0+S_1} l_i p\,dS - \frac{1}{4\pi c^2}\frac{1}{|\mathbf{x}|}\frac{\partial}{\partial t}\int_{S_1} l_i(\rho_0 u_i)\,dS. \quad (4.7)$$

Thus the theory predicts that the acoustic field is generated by compact dipole and monopole sources acting at the co-ordinate origin.

Before investigating the source strengths more closely we consider briefly another form of solution to Lighthill's equation (4.3), derived by Powell (1964) in his theory of vortex sound. Under the same assumptions as were used to obtain (4.7), Powell's solution can be written as

$$\rho(\mathbf{x},t)-\rho_0 = \frac{1}{4\pi c^2}\frac{x_i}{|\mathbf{x}|^2}\frac{\partial}{\partial t}\int_{S_1+S_0} l_i(p+\tfrac{1}{2}\rho_0 u^2)\,dS - \frac{1}{4\pi c^2}\frac{1}{|\mathbf{x}|}\frac{\partial}{\partial t}\int_{S_1} l_i\rho_0 u_i\,dS + \frac{1}{4\pi c^2}\frac{x_i}{|\mathbf{x}|^2}\frac{\partial}{\partial t}\int_{V'}\rho_0(\boldsymbol{\omega}\times\mathbf{u})_i\,dV. \quad (4.8)$$

On S_1, $\frac{1}{2}\rho_0 u^2$ is of $O(M^2)$ and can be neglected. Since we have a delta-function distribution of vorticity the volume integral in (4.8) takes a particularly simple form; in fact, the axial component of

$$\int_{V'}\rho_0\boldsymbol{\omega}\times\mathbf{u}\,dV \quad (4.9)$$

is $2\rho_0 K\,\partial y_0/\partial t$, which is equivalent to a force acting on the fluid in the positive-x_1 direction. It is well known that the potential flow of an inviscid incompressible fluid round a sharp edge induces a suction (Batchelor 1967, p. 412). By integrating the pressure round the 'streamlines' S_0, which we subsequently allow to collapse onto each wall of the channel, we can determine this 'edge force'. However, since the suction arises from the $r^{-\frac{1}{2}}$ velocity singularity at the edge, we find that

$$\int_{S_0} l_i(p+\tfrac{1}{2}\rho_0 u^2)\,dS = -\int_{S_\infty} l_i\frac{\partial}{\partial t}(\rho_0\phi)\,dS = 0. \quad (4.10)$$

On substituting for p and u_i on S_1 we find that the surface integrals of (4.8) produce an axial dipole of strength $-2\rho_0 K\,\partial\eta_0/\partial t$ per unit volume and a weak monopole of strength $-2\rho_0 KM\,\partial\eta_0/\partial t$ per unit volume. Thus by employing Powell's result we have recovered the solution obtained by matching, and furthermore we see that the distribution of axial dipole sources on the exit plane drives a monopole-producing mass efflux from the duct, and these two sources supplement the apparent dipole that Powell recognized at the vortex centres.

Returning to Curle's solution we see that it differs from Powell's formulation in replacing the volume integral of $\rho_0 \boldsymbol{\omega}\times\mathbf{u}$ by a surface integral of the pressure over S_0, which we call the 'edge force'. We now show that these two terms are in fact identical to first order in M. The 'edge force' F_i can be written as

$$F_i = \int_{S_0} l_j(p\delta_{ij}+\rho_0 u_i u_j)\,dS(\mathrm{y}) \tag{4.11}$$

$$= \int_{S_0+S_1+S_\infty+S^*} l_j(p\delta_{ij}+\rho_0 u_i u_j)\,dS - \int_{S_1+S_\infty+S^*} l_j(p\delta_{ij}+\rho_0 u_i u_j)\,dS. \tag{4.12}$$

Applying the divergence theorem and using (4.2) gives

$$F_i = -\int_{V_1}\frac{\partial}{\partial t}(\rho_0 u_i)\,dV - \int_{S_1+S_\infty} l_j(p\delta_{ij}+\rho_0 u_i u_j)\,dS - \int_{V^*}\frac{\partial}{\partial t}(\rho_0 u_i)\,dV \tag{4.13}$$

$$= -\int_{V_1}\frac{\partial}{\partial y_i}\frac{\partial}{\partial t}(\rho_0\phi)\,dV - \int_{S_1+S_\infty} l_j(p\delta_{ij}+\rho_0 u_i u_j)\,dS - \int_{V^*}\frac{\partial}{\partial t}(\rho_0 u_i)\,dV. \tag{4.14}$$

The volume V' has here been divided into two parts (see figure 3). The first, V_1, contains irrotational flow only, while the second, V^*, bounded by the surface S^* of radius ϵ, contains vorticity. In the limit as $\epsilon\to 0$, the contribution from the volume integral over V^* is of $O(\epsilon)$ and so

$$F_i = -\int_{S_0+S_1+S_\infty+S^*} l_i\frac{\partial\rho_0\phi}{\partial t}\,dS - \int_{S_1+S_\infty} l_j(p\delta_{ij}+\rho_0 u_i u_j)\,dS. \tag{4.15}$$

On S_1 and S_∞, quadratic terms in u_i can be neglected and $p = -\partial(\rho_0\phi)/\partial t$. Hence

$$F_i = -\rho_0\int_{S^*} l_i\frac{\partial}{\partial t}(\rho_0\phi)\,dS. \tag{4.16}$$

On S^*, however, we can write $p+\tfrac{1}{2}\rho_0 u^2 = -\partial(\rho_0\phi)/\partial t$, and therefore

$$F_i = -\int_{V^*}\frac{\partial}{\partial y_i}(p+\tfrac{1}{2}\rho_0 u^2)\,dV \tag{4.17}$$

$$= \rho_0\int_{V^*}\frac{\partial u_i}{\partial t}\,dV + \rho_0\int_{V^*}(\boldsymbol{\omega}\times\mathbf{u})_i\,dV. \tag{4.18}$$

Again as $\epsilon\to 0$, the first integral tends to zero, and finally we have that

$$F_i = \rho_0\int_{V^*}(\boldsymbol{\omega}\times\mathbf{u})_i\,dV. \tag{4.19}$$

That the 'edge force' is equal to the 'vortex force' (4.19) can also be shown by direct evaluation of the edge force. Following the method of Batchelor we determine the force in the axial direction to be

$$A(t) = F_1 = \tfrac{1}{2}\rho_0 A^2\pi, \tag{4.20}$$

where
$$A(t) = \frac{K}{2\pi}\frac{g(t)}{(1+e^{f(Z_0)})\,(1+e^{\overline{f(Z_0)}})} \tag{4.21}$$

in the notation of §§ 2 and 3. By using the formula for $\partial y/\partial t$ given by (2.9), it is a relatively straightforward matter to show that the edge force F_1 is indeed equal in magnitude to the so-called vortex force.

5. Motion of a single vortex

To determine the radiation induced by the motion of a single vortex coupled to a semi-infinite duct we employ the analytic scheme of §§ 2 and 3 again. To reduce repetition, the results of each stage in the calculation are stated quite briefly.

Suppose that an infinite channel is defined in the complex-ξ plane as before, and consider a single negative line vortex filament, strength K, at

$$\xi = \xi_0 + i(\pi - \delta).$$

The complex potential is given by

$$\Omega(\xi) = \frac{iK}{2\pi}\log_e\left\{\frac{\sin[(\xi-\xi_0)/4i]}{\sin[(\xi-\xi_0-2i\delta)/4i]}\right\} + \gamma(t). \tag{5.1}$$

Transforming to the complex-Z plane we obtain

$$\Omega(Z) = \phi + i\psi = \frac{iK}{2\pi}\log_e\left\{\frac{\sin[(f(Z)-f(Z_0))/4i]}{\sin[(f(Z)-f(Z_0)-2i\nu)/4i]}\right\} + \gamma(t), \tag{5.2}$$

where

$$\nu(t) = \pi - \operatorname{Im} f(Z_0). \tag{5.3}$$

In addition we have

$$\frac{d\Omega}{dZ} = u - iv = \frac{K}{8\pi}\left(\frac{1}{1+e^{f(Z)}}\right)\left\{\cot\frac{1}{4i}(f(Z)-f(Z_0)) - \cot\frac{1}{4i}(f(Z)-f(Z_0)-2i\nu)\right\} \tag{5.4}$$

and expanding (5.4) about Z_0 we obtain a differential relation governing the vortex path:

$$\frac{dx_0}{dt} - i\frac{dy_0}{dt} = \frac{K}{8\pi}\left(\frac{1}{1+e^{f(Z_0)}}\right)\left\{\frac{2i\,e^{f(Z_0)}}{1+e^{f(Z_0)}} - \cot\tfrac{1}{2}\nu\right\}. \tag{5.5}$$

The paths obtained by numerical integration for different values of δ show that the path of a single vortex is similar to that shown in figure 1, for an individual member of a vortex pair near the channel wall. In this case, however, zero propagation velocity occurs when $\delta = \pi$; thus for $0 < \delta < \pi$ a vortex starting inside the duct will always return to $x = -\infty$ outside the duct. As $x \to -\infty$, the vortex speed inside the duct tends to $(K/8\pi)\cot\frac{1}{2}\delta$, and outside the duct to $K/4\pi(\operatorname{Im} Z_0 - f(Z_0))$.

Since the vortex motion exhibits a rapid change near the duct exit, we can formulate the singular perturbation problem in the same way as in § 3. The main difference in the analysis is that we only need to match the 'inner' hydrodynamic solution to an exterior two-dimensional wave field, there being no induced flow from the duct.

The potential solution to the 'inner' problem can be written in non-dimensional space–time variables as

$$\phi = \operatorname{Re}\Omega(Z) = \operatorname{Re}\left\{\frac{iK}{2\pi}\log_e\left[\frac{\sin[(f(Z)-f(Z_0))/4i]}{\sin[(f(Z)-f(Z_0)-2i\nu)/4i]}\right]\right\} + \operatorname{Re}\gamma(\tau). \tag{5.6}$$

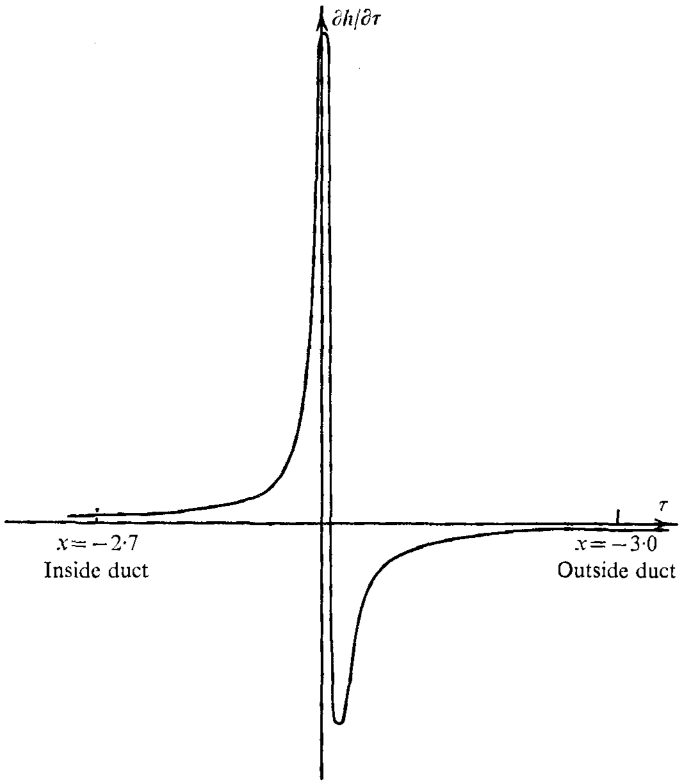

FIGURE 4. Time variation of $\partial h/\partial \tau$: $\delta \simeq 0{\cdot}3$.

Writing ϕ in 'outer' variables and letting $M \to 0$ we obtain

$$\phi\left(\frac{x'}{M},\frac{y'}{M},\tau\right) = h(\tau)\sin\frac{\theta}{2}\frac{M^{\frac{1}{2}}}{r^{\frac{1}{2}}} + \operatorname{Re}\gamma(\tau) - \frac{K\nu(\tau)}{4\pi} + O(M), \tag{5.7}$$

with

$$h(\tau) = -2e^{\frac{1}{2}\xi_0}\cos\tfrac{1}{2}\eta_0. \tag{5.8}$$

Let

$$\phi_0(\mathbf{x}',\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\varpi\tau}\,\phi_0^*(\mathbf{x}',\varpi)\,d\varpi \tag{5.9}$$

then ϕ_0^* satisfies the two-dimensional homogeneous Helmholtz equation

$$\frac{\partial^2\phi_0^*}{\partial x'^2} + \frac{\partial^2\phi_0^*}{\partial y'^2} + \varpi^2\phi_0^* = 0. \tag{5.10}$$

Solving for ϕ_0^* subject to the appropriate transformed boundary conditions, we obtain

$$\phi_0^*(\mathbf{x}',\varpi) = A(\varpi)\,M^{\frac{1}{2}}H_{\frac{1}{2}}^{(1)}(\varpi r)\sin\tfrac{1}{2}\theta. \tag{5.11}$$

To satisfy the radiation condition at infinity we have to put $\operatorname{Re}\gamma(\tau) = K\nu(\tau)/4\pi$, and the coefficient $A(\varpi)$ is determined through matching ϕ_0^* to the Fourier transform of the 'inner' potential ϕ:

$$A(\varpi) = \frac{iK}{2\pi}\left(\frac{\pi\varpi}{2}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} h(\tau)\,e^{i\varpi\tau}\,d\tau. \tag{5.12}$$

Again, using simple generalized function theory and the results (5.11) and (5.12), it is possible to derive an equation for E, the total energy per unit length radiated to infinity in the sound field throughout all time:

$$E = \frac{\rho_0 M K^2}{4\pi^2} \int_{-\infty}^{\infty} \left(\frac{\partial h}{\partial \tau}\right)^2 d\tau. \tag{5.13}$$

Since the velocity scale U depends on the vortex strength K, we see that E is proportional to K^3, a result typical of two-dimensional edge scattering problems. Figure 4 illustrates the time behaviour of $\partial h/\partial \tau$ and shows clearly that almost all the energy radiated as sound originates while the vortex is close to the duct exit.

In the limit as $\delta \rightarrow 0$ the vortex path tends to the path followed by a single vortex coupled to a half-plane. Using Crighton's solution for the value of the coefficient of the edge singularity in the latter problem, we can compute corresponding values of E. With $\delta = 0{\cdot}1$ the ratio of the energy scattered by the duct to the energy scattered by a half-plane is 0·56. This result compares quite favourably with the theoretical prediction of Crighton & Leppington (1971) that the ratio of duct and half-plane efficiencies should be 0·5. Evaluating

$$\int_{-\infty}^{\infty} \left(\frac{\partial h}{\partial \tau}\right)^2 d\tau$$

for various values of δ, we obtain numerical results which change slowly from 0·19 when $\delta = 0{\cdot}01$ to 0·3 when $\delta = 2{\cdot}65$ and then fall to zero at $\delta = \pi$.

6. Summary

We have discovered that the sound fields obtained when vortices exhaust from two-dimensional ducts fall into two distinct classes which correspond to the presence or absence of symmetry in the unsteady flow. In the latter case the solution differs only in the coefficient of the edge singularity from the analogous result for scattering from a single half-plane. However, when the edge scattering effects cancel we find that the velocity singularities at the edges of the duct still play a novel and fundamental role in determining the acoustic field. It is perhaps surprising that the general theories of Curle and Powell should be related through the identity of the 'edge' and 'vortex' forces. It is perhaps also surprising that no difficulties arise over the slow decay of the velocity field as $r \rightarrow \infty$ in the hydrodynamic flow. These could result in the problem being improperly posed in the usual sense of being able to separate the regions of sound generation and wave propagation. The successful matching (of the inner and outer flows as well as Lighthill's theory with the singular perturbation scheme) seems to allay these doubts and justifies the steps made in § 4, where the sound field is supposed to be generated by a compact source region.

One of us (P. C.) acknowledges the support of an S.R.C. research studentship.

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